

On the automorphism group of the asymptotic pants complex of a planar surface of infinite type

Ariadna Fossas Maxime Nguyen
 Institut Fourier
 Université de Grenoble I
 38402 St. Martin d'Hères. France
 ariadna.fossas@ujf-grenoble.fr
 maxime.nguyen@ujf-grenoble.fr

Abstract

We consider a planar surface Σ of infinite type which has the Thompson group \mathcal{T} as asymptotic mapping class group. We construct the asymptotic pants complex \mathcal{C} of Σ and prove that the group \mathcal{T} acts transitively by automorphisms on it. Finally, we establish that the automorphism group of the complex \mathcal{C} is an extension of the Thompson group \mathcal{T} by $\mathbb{Z}/2\mathbb{Z}$.

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1 Introduction

Given a compact surface of genus g with b boundary components, one can build various locally infinite simplicial complexes on which the mapping class group of the surface acts by automorphisms, mainly the curve complex, the arc complex and the pants complex. The automorphism group of each one of these complexes is related to the mapping class group itself. Specifically, Ivanov in [14] and subsequently Luo in [19] proved that the automorphism group of the curve complex is the extended mapping class group for almost all compact surfaces, and Korkmaz extended this result to punctured spheres and punctured tori in [15]. Analogous results were proved for the arc complex by Ivanov [14] and Irmak and McCarthy [13], for the pants complex by Margalit [20], for the Hatcher-Thurston complex by Irmak and Korkmaz [12], for the arc and curve complex by Korkmaz and Papadopoulos [16], and for the complex of non-separating curves by Schmutz-Schaller [21] and subsequently by Irmak [11].

When dealing with surfaces of infinite type, one issue is to find good analogues for mapping class groups and for the complexes enumerated above. Funar and Kapoudjan defined the notion of asymptotic mapping class group for some surface of genus zero and infinite type in [6], the first one of a series of three papers (see also [7] and [8]). They constructed several complexes for this type of surfaces and proved that the asymptotic mapping class group acts on them by automorphisms. These complexes can be seen as generalizations of the pants complex to the infinite type case. It is worth to mention that the asymptotic mapping class groups they defined are closely related to Thompson's group \mathcal{T} , which is one of the first examples of finitely presented infinite simple groups.

In this paper, we consider the case of a planar infinite surface with non compact boundary, which has the Thompson group \mathcal{T} as its asymptotic mapping class group. Then, we construct a locally infinite cellular complex \mathcal{C} related to the pants complex of the infinite surface of genus zero of [6]. We prove that the group \mathcal{T} acts by automorphisms on the complex and, finally, we establish that the automorphism group of the complex is $\mathcal{T} \rtimes \mathbb{Z}/2\mathbb{Z}$. We also give a geometric interpretation to this semi-direct product as the extended asymptotic mapping class group, obtaining in this way a complete analogue to Ivanov's

theorem for the surface Σ of infinite type.

Our result can also be considered from a different viewpoint. In 1970, Tits proved in [23] that most subgroups of automorphism groups of trees generated by vertex stabilizers are simple. This result was extended by Haglund and Paulin in [9] to the case of automorphism groups of negatively curved polyhedral complexes which are (in general uncountable) non-linear and virtually simple. On the other hand, Farley proved in [5] that the Thompson groups \mathcal{F} , \mathcal{T} and \mathcal{V} act properly and isometrically on CAT(0) cubical complexes. In particular, these groups are a-T-menable and satisfy the Baum-Connes conjecture with arbitrary coefficients. This leads to the legitimate question of realizing the simple groups \mathcal{T} and \mathcal{V} as automorphisms groups of suitable cell complexes. Although the complex which we obtain is not Gromov hyperbolic, it gives a convenient answer to this question for the group \mathcal{T} .

Definitions and statement of the results

The surface Σ and its asymptotic mapping class group

Let \mathbb{D}^2 be the hyperbolic disc and suppose that its boundary $\partial\mathbb{D}^2$ is parametrized by the unit interval (with identified endpoints). Let E denote the triangulation of \mathbb{D}^2 given by the family of bi-infinite geodesics representing the standard dyadic intervals, i.e. the family of geodesics I_a^n joining the points $p = \frac{a}{2^n}$, $q = \frac{a+1}{2^n}$ on $\partial\mathbb{D}^2$, where a, n are integers satisfying $0 \leq a \leq 2^n - 1$. Let T_3 be the dual graph of E , which is an infinite (unrooted) trivalent tree. We choose to realize the tree T_3 in \mathbb{D}^2 by using geodesic segments obtained from the I_a^n (with $n \geq 1$ and a even) by rotations of angles $\frac{\pi}{2^n}$, as shown in figure 1. We also denote by T_3 this realisation.

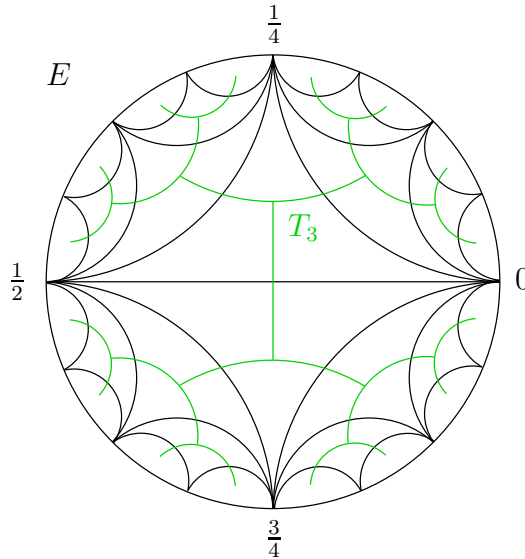


Figure 1: The triangulation E of \mathbb{D}^2 and its dual tree T_3 .

Now, let Σ be a closed δ -neighbourhood of T_3 (see figure 2). Remark that the surface Σ is planar (thus oriented), non compact, contractible and its boundary has infinitely many connected components. Note that we can choose δ small enough to avoid self-intersections of the boundary and such that each intersection $I_a^n \cap \Sigma$ is a single connected arc joining two boundary components of Σ .

The triangulation E together with the boundary components of Σ define a tessellation of our surface into hexagons (see figure 2). By abuse of notation we will denote this hexagonal tessellation also by E . Note that the boundary of each hexagon alternates sides contained in different connected components of $\partial\Sigma$ with sides contained in different arcs of the triangulation E .

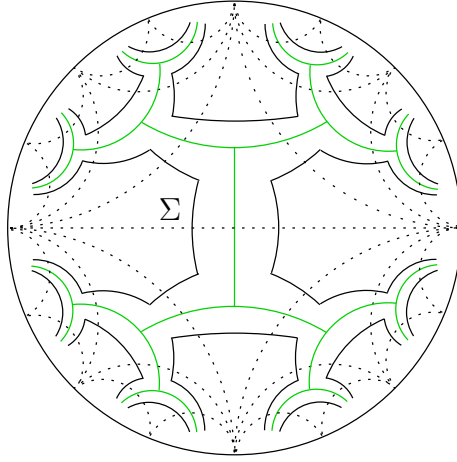


Figure 2: The surface Σ and its hexagonal tessellation.

Definition 1. The set of arcs $\{I_{a_1}^{n_1}, \dots, I_{a_k}^{n_k}\}$ of E is separating if the union of its arcs bounds a compact subsurface of Σ containing $I_0^1 \cap \Sigma$.

Let $\text{Homeo}(\Sigma)$ denote the group of orientation-preserving homeomorphisms of Σ .

Definition 2. An element $\varphi \in \text{Homeo}(\Sigma)$ is asymptotically rigid if there exists a set of separating arcs $\{I_{a_1}^{n_1}, \dots, I_{a_k}^{n_k}\}$ such that:

1. For all $1 \leq i \leq k$, there exist $I_{b_i}^{m_i}$ such that $\varphi(I_{a_i}^{n_i}) \cap \Sigma = I_{b_i}^{m_i} \cap \Sigma$, and
2. For all $1 \leq i \leq k$, for all $j \in \mathbb{N}$, and for all $0 \leq l < 2^j$ the arc $\varphi(I_{2^j a_i + l}^{n_i + j}) \cap \Sigma$ is equal to the arc $I_{2^j b_i + l}^{m_i + j} \cap \Sigma$.

Remark 1. The set of arcs $\{I_{b_1}^{m_1}, \dots, I_{b_k}^{m_k}\}$ from the definition above is a set of separating arcs.

Remark 2. Let φ be an asymptotically rigid homeomorphism. The first condition of the definition says that there exists a compact subsurface $C \subset \Sigma$ which boundary is composed by segments of $\partial\Sigma$ and arcs of E such that φ sends C to another compact subsurface C' of the same type (i.e. which boundary is composed by segments of $\partial\Sigma$ and arcs of E). The second condition deals with the behaviour of φ outside C . Let t_1, \dots, t_k be the connected components of $T_3 \setminus (T_3 \cap C)$ and t'_1, \dots, t'_k be the connected components of $T_3 \setminus (T_3 \cap C')$. Then the second condition says that each t_i is 'naturally' sent to t'_i .

The group of asymptotically rigid homeomorphisms of Σ will be denoted by $\text{Homeo}_a(\Sigma)$.

Definition 3. The asymptotic mapping class group of Σ is the quotient of $\text{Homeo}_a(\Sigma)$ by the group of isotopies of Σ .

Remark 3. ([7], section 1.3) The asymptotic mapping class group of Σ is isomorphic to the Thompson group \mathcal{T} , which is the group of orientation preserving piecewise linear homeomorphisms of the circle that are differentiable except at finitely many dyadic rational numbers, such that, on intervals of differentiability, the derivatives are powers of 2, and which send the set of dyadic rational numbers to itself. For a complete introduction to Thompson's groups see [3].

The complex \mathcal{C}

Once the surface Σ defined, we are going to define a 2-dimensional cellular complex \mathcal{C} . The vertices of \mathcal{C} are the isotopy classes relatively to the boundary of the tessellations of Σ into hexagons which differ from the distinguished tessellation E only in a finite number of hexagons. Note that every hexagon of a

tessellation have 3 sides corresponding to different boundary components of Σ and 3 sides corresponding to arcs joining these boundary components.

Let v be a vertex of \mathcal{C} , and let a be an arc bounding two hexagons h_1, h_2 of the tessellation v . Let a' be an arc contained in $h_1 \cup h_2$ joining two boundary components of Σ such that $i([a], [a']) = 1$, where i denote the geometric intersection number and $[a]$ the isotopy class of a relatively to the boundary. Let w be the vertex of \mathcal{C} obtained from v by replacing the arc a by a' . Then, the vertices v and w are joined by an edge on \mathcal{C} . We say that w is obtained from v by *flipping* the arc a .

Let v be a vertex of \mathcal{C} and let a and b be two arcs, each one of which bounding two hexagons of the tessellation v . We say that a and b are *adjacent* if there exists an hexagon on the tessellation v having both of them on its boundary. There are two possible cases:

1. The two arcs are not adjacent. Then, flipping first a and then b one obtains the same tessellation as the result of first flipping b and then a . Thus, the commutativity of flips defines a 4-cycle on the 1-skeleton of \mathcal{C} and we fill it with a *squared 2-cell*.
2. The two arcs are adjacent. Then, flipping first a and then b one obtains a different tessellation from the result of first flipping b and then a . These two tessellations differ from a flip. We fill the corresponding 5-cycle with a *pentagonal 2-cell*.

We obtain, in particular, the following analogue to the case of the cellular complexes associated to compact surfaces:

Proposition 4. The asymptotic mapping class group of Σ acts transitively on the complex \mathcal{C} by automorphisms.

It is now natural to ask whether or not the automorphism group of the cellular complex \mathcal{C} is related to the asymptotic mapping class group of the surface Σ . To answer this question, we construct the following exact sequence:

$$1 \longrightarrow \mathcal{T} \longrightarrow \text{Aut}(\mathcal{C}) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1,$$

and prove that it splits.

Thus, we obtain the following result:

Theorem 1. The automorphism group of the complex \mathcal{C} is isomorphic to the semi-direct product $\mathcal{T} \rtimes \mathbb{Z}/2\mathbb{Z}$.

Furthermore, the extension is not trivial; we obtain a non trivial action of $\mathbb{Z}/2\mathbb{Z}$ to $\text{Out}(\mathcal{T})$. Thus, as Brin established in [2] that the group of outer automorphisms of \mathcal{T} is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, we get an isomorphism between $\text{Out}(\mathcal{T})$ and $\mathbb{Z}/2\mathbb{Z}$.

Finally, the tools used to prove the main theorem lead us to give a geometric interpretation of this result. It turns out that one can classify the elements of $\text{Aut}(\mathcal{C})$ in two categories: those which preserve (in some natural sense) the orientation of the complex, and those which reverse the orientation. Then, it is natural to define the *extended mapping class group* of Σ as $\mathcal{T} \rtimes \mathbb{Z}/2\mathbb{Z}$, which makes theorem 1 a complete analogue to Ivanov's theorem for compact surfaces.

Structure of the paper

This paper is structured as follows. In the second section, we provide a modification of the surface Σ which does not involve changes on the mapping class group. This will allow us to describe the isomorphism between the asymptotic mapping class group and Thompson's group \mathcal{T} . In the third section, we briefly discuss the analogy between the complex \mathcal{C} and the complexes obtained for compact surfaces and define the action of the asymptotic mapping class group on \mathcal{C} . Finally, in the fourth section, we describe the structure of the automorphisms group of the complex \mathcal{C} , proving theorem 1.

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2 The surface and its asymptotic mapping class group

In this section we introduce a slightly different surface Σ' which allows us to simplify the situation. Both surfaces Σ and Σ' have the same asymptotic mapping class group.

Definition 4. *The surface Σ' is obtained by attaching to the hyperbolic disk \mathbb{D}^2 the points of its boundary corresponding to the dyadic rational numbers.*

Remark 4. The triangulation E of \mathbb{D}^2 is a triangulation of Σ' having for vertices all the boundary components of Σ' .

Lemma 1. *Let $\partial_0\Sigma$ be a connected component of the boundary of Σ , which is a bi-infinite line on \mathbb{D}^2 . The adherence of $\partial_0\Sigma$ is $\partial_0\Sigma \cup \{q\}$, where q is a dyadic rational number on $\partial\mathbb{D}^2$. Thus, there is a natural bijection between the set of dyadic rational numbers $\partial\Sigma'$ and the set of connected components of the boundary of Σ .*

Corollary 1. *Let $\partial_0\Sigma$ be a connected component of the boundary of Σ , and let $I_{a_1}^{n_1}, I_{a_2}^{n_2}$ be two geodesics of E intersecting $\partial_0\Sigma$. Then, $I_{a_1}^{n_1}$ and $I_{a_2}^{n_2}$ have a common boundary point (dyadic).*

Corollary 2. *Let $\{I_{a_1}^{n_1}, \dots, I_{a_k}^{n_k}\}$ be a set of separating arcs which are cyclically ordered. Then, they are the sides of an ideal hyperbolic k -gon.*

Corollary 3. *One can deform continuously Σ to Σ' by an homotopy $H : \Sigma \times [0, 1] \rightarrow \mathbb{D}^2 \cup \partial\mathbb{D}^2$ such that:*

- *For $t < 1$, $H(\cdot, t)$ is an homeomorphism.*
- *For $t < 1$, $H(\Sigma \cap I_a^n, t) \subset I_a^n$.*
- *$H(\cdot, 1)$ is an homeomorphism between the interior of Σ and \mathbb{D}^2 .*
- *Let $\partial_0\Sigma$ be a connected component of the boundary of Σ and let q be the dyadic rational number on its adherence. Then, $H(\partial_0\Sigma, 1) = q$.*

One can now define the notion of asymptotically rigid homeomorphism of Σ' , and the asymptotic mapping class group by taking $\text{Homeo}_a(\Sigma')$ modulo isotopies of Σ' .

Proposition 1. *The asymptotic mapping class groups of Σ and Σ' are isomorphic.*

Furthermore, they are isomorphic to Thompson’s group \mathcal{T} .

Proof. Let $\varphi \in \text{Homeo}_a(\Sigma)$. We define $\varphi' \in \text{Homeo}_a(\Sigma')$ as follows:

$$\varphi'(x) = \begin{cases} H(\varphi(y), 1), & \text{if } x \in \mathbb{D}^2 \\ H(\varphi(\partial_x\Sigma), 1), & \text{if } x \in \partial\Sigma', \end{cases}$$

where $y \in \Sigma$ such that $H(y, 1) = x$, and $\partial_x\Sigma$ is the connected component of $\partial\Sigma$ having x on its adherence and H is the homotopy from lemma 3. Note that every isotopy of Σ can be extended to an isotopy of $\mathbb{D}^2 \cup \partial\mathbb{D}^2$. Thus, the application we defined induces an injective morphism for the mapping class groups.

Reciprocally, let $[\varphi']$ be an element of the asymptotic mapping class group of Σ' . Note that we can choose a representative $\varphi' \in \text{Homeo}_a(\Sigma')$ such that $\varphi'_\Sigma \in \text{Homeo}_a(\Sigma)$. Thus, we defined an injective morphism from the asymptotic mapping class group of Σ' to the asymptotic mapping class group of Σ .

Finally, one can verify that the composition of both is the identity. □

Before giving the isomorphism between \mathcal{T} and the asymptotic mapping class group of Σ , let us give the analytical definition of Thompson's group \mathcal{T} :

Definition 5. *The Thompson group \mathcal{T} is the set of orientation preserving piecewise linear homeomorphisms of the circle $S^1 = [0, 1]/_{0 \sim 1}$ whose points of non-differentiability are dyadic rational numbers, whose derivatives (where defined) are powers of 2, which send the set of dyadic rational numbers to itself.*

Let φ be a representative of an equivalence classe of the asymptotic mapping class group of Σ' . We claim that φ acts as an element of the Thompson group \mathcal{T} on $\partial\Sigma'$, which is the set of dyadic rational numbers, and that this action does not depend on the representative. To see this, let $\{I_{a_1}^{n_1}, \dots, I_{a_k}^{n_k}\}$ and $\{I_{b_1}^{m_1}, \dots, I_{b_k}^{m_k}\}$ be sets of separating arcs associated to φ , with $\varphi(I_{a_i}^{n_i}) = I_{b_i}^{m_i}$ for $1 \leq i \leq k$. Without loss of generality, we can suppose that $a_1 = 0$, $a_k = 2^{n_k} - 1$ (I_a^1 is contained on the compact subsurface bounded by the arcs I_a^n) and that both sets are cyclically ordered. Let f be the unique map of the circle satisfying the following conditions:

1. For all $1 \leq i \leq k$, $f\left(\frac{a_i}{2^{n_i}}\right) = \frac{b_i}{2^{m_i}}$,
2. For all $1 \leq i \leq k$, $f\left(\frac{a_i + 1}{2^{n_i}}\right) = \frac{b_i + 1}{2^{m_i}}$, and
3. For all $x \in \left(\frac{a_i}{2^{n_i}}, \frac{a_i + 1}{2^{n_i}}\right)$, $f'(x) = 2^{n_i - m_i}$.

One can see that $f \in \mathcal{T}$ using the corollary 2 and the fact that φ is orientation-preserving.

Reciprocally, given an element $f \in \mathcal{T}$ it can be proven (adaptation to \mathcal{T} of lemma 2.2 from [3]) that there exists a partition of the unit interval into standard dyadic intervals $0 = x_0 < x_1 < \dots < x_k = 1$, and an integer $0 \leq j \leq k$ such that:

1. The subintervals of the partition $0 = f(x_j) < f(x_{j+1}) < \dots < f(x_k) = f(x_0) < \dots < f(x_{j-1}) < f(x_j) + 1 = 1$ are standard dyadic, and
2. The map f is linear in every subinterval of the partition.

Thus, we can find two ideal k -gons inscribed in E defining an element φ in the asymptotic mapping class group of Σ' which acts as f on $\partial\Sigma'$. This ends the proof of the following result (see [7]):

Proposition 2. *The asymptotic mapping class group of Σ is isomorphic to the Thompson group \mathcal{T} .*

For the purposes of this paper it is useful to think of the Thompson group \mathcal{T} as the group generated by two elements: an element α of order 4 and an element β of order 3. The full presentation is the following (for the details see [7] and [18]):

$$\mathcal{T} \simeq \langle \alpha, \beta \mid \alpha^4, \beta^3, [\beta\alpha\beta, \alpha^2\beta\alpha\beta\alpha^2], [\beta\alpha\beta, \alpha^2\beta^2\alpha^2\beta\alpha\beta\alpha^2\beta\alpha^2], (\beta\alpha)^5 \rangle.$$

The two generators α, β of Thompson's group \mathcal{T} can be defined in terms of polygons as follows: α sends $\{I_0^2, I_1^2, I_2^2, I_3^2\}$ respectively to $\{I_1^2, I_2^2, I_3^2, I_0^2\}$ and β sends $\{I_0^1, I_0^2, I_1^2\}$ respectively to $\{I_0^2, I_1^2, I_0^1\}$.

Remark that the polygons defining an element of \mathcal{T} are not unique, although there is a minimal one which satisfies the conditions. Consider this minimal pair of polygons defining an element $f \in \mathcal{T}$. Then, if we split a standard dyadic interval of the source polygon in two halves and we also split the standard dyadic interval corresponding to its image in two halves, the element defined by the new pair of polygons is the same. All possible pairs of polygons defining f are obtained doing this expanding operation finitely many times.

3 The asymptotic pants complex

First of all, we would like to present the relation of the complex \mathcal{C} and the different known complexes for the compact case. In some general sense, one can see \mathcal{C} as an analogue to the pants complex. For this, take another copy of the initial surface Σ , already tessellated in hexagons by E , and glue the two copies together along their boundary. Now, the arcs of E on the two copies are also glued and they become simple curves which decompose the doubled surface in infinitely many pairs of pants. Hence, the vertices of the complex \mathcal{C} can be seen as decompositions in pairs of pants of the double surface, obtained from the initial decomposition by a finite number of elementary moves. An elementary move consists on changing a simple curve γ by another γ' , which does not intersect the other curves of the pants decomposition, and with minimal geometric intersection number $i([\gamma], [\gamma']) = 2$. In our case, we only consider the elementary moves that can be seen on the planar surface Σ . For the details of this construction see [6] and [7].

The rest of this section is devoted to give another point of view of the complex \mathcal{C} using the definition of Σ' . For example, we can see the 0-skeleton of \mathcal{C} as the set of triangulations of Σ' with vertices in $\partial\Sigma'$ which differ from E only on a finite number of triangles. Hence, two vertices u, v of \mathcal{C} are joined by an edge if we can obtain v from u by changing the diagonal of a quadrilateral inscribed on the triangulation u . We will that v is obtained by *flipping an arc of u* .

Remark 5. (Geometric interpretation of the 2-cells) Each 2-cell of the complex can be seen, geometrically, as the object obtained from a triangulation v of the complex \mathcal{C} by flipping consecutively two of its arcs. If there exists a triangle in v containing the two arcs on its boundary, then the 2-cell is pentagonal. Otherwise, the 2-cell is squared.

Remark 6. Given three vertices u, v, w of \mathcal{C} where u, w are different and adjacent to v , there exists a unique 2-cell containing u, v, w . This follows from the precedent remark.

Definition 6. We call the 2-dimensional cellular complex \mathcal{C} the asymptotic pants complex of the surfaces Σ and Σ' .

It is worth to mention that the asymptotic pants complex is closely related to the graph of triangulations of a convex n -gon (see, for example, [1], [17] and [10]) and hence, it is also related to the rotation distance in binary trees (see [22]). The graph of triangulations of a convex n -gon is a graph whose vertices are the triangulations of the n -gon in $n - 2$ triangles, and where two vertices are joined by an edge if there is a single flip going from one to the other. Thus, the asymptotic pants complex can be seen as a 2-dimensional complex associated to the triangulations graph of an infinite-sided convex polygon. Now, one can derive the connectivity of the complex \mathcal{C} from the finite case (see [10] for a simple proof). To see this, one can think that every vertex of \mathcal{C} has only a finite number of arcs not contained in the triangulation E and so, there exists a convex polygon having all of this arcs on its interior. Furthermore, Lee in [17] proved that the complex of triangulations of a convex n -gon is a convex polytope of dimension $n - 3$ (see also [4]). Thus, the complex \mathcal{C} is also simply connected.

Proposition 3. The complex \mathcal{C} is connected and simply connected.

Finally, we will define the action of \mathcal{T} on \mathcal{C} . Let v be a vertex of \mathcal{C} and let f be an element of \mathcal{T} . Consider the set of geodesics G defined as follows: the geodesic joining the dyadic rational numbers p, q is in G if the geodesic joining the dyadic rational numbers $f^{-1}(p)$ and $f^{-1}(q)$ belongs to the triangulation v . Remark that the geodesics on G define a triangulation of Σ' ; this triangulation will be the image by f of the vertex v and we will denote it by $f \cdot v$. As a consequence of the definitions of \mathcal{T} and \mathcal{C} , the triangulation $f \cdot v$ corresponds to a vertex of \mathcal{C} . Note that it is possible to find an ideal polygon bounded by the set of separating arcs $\{I_{a_1}^{n_1}, \dots, I_{a_k}^{n_k}\}$ which is the source polygon for the element f and which contains all the geodesics that are on the triangulation v and which are not on the base triangulation E .

Let v, w be two vertices joined by an edge in \mathcal{C} , and let f be an element of \mathcal{T} . It follows from the definition of the action of f on the vertices of \mathcal{C} that $f \cdot v$ and $f \cdot w$ are joined by an edge in \mathcal{C} . It also follows from the definition of the action on the vertices together with the geometric interpretation of the 2-cells (remark 5) that the action can be extended to the 2-skeleton of \mathcal{C} . Thus, we can define a natural map $\Psi : \mathcal{T} \rightarrow \text{Aut}(\mathcal{C})$.

Proposition 4. *The asymptotic mapping class group \mathcal{T} of Σ' acts transitively on the asymptotic pants complex \mathcal{C} by automorphisms. Furthermore, the map $\Psi : \mathcal{T} \rightarrow \text{Aut}(\mathcal{C})$ is injective.*

Proof. It follows from the paragraph preceding the proposition that \mathcal{T} acts on \mathcal{C} by automorphisms. We prove the transitivity by showing that given a vertex v of \mathcal{C} there exists an element $f \in \mathcal{T}$ such that $f \cdot E = v$. Let P be an ideal polygon inscribed on E containing all the edges of the triangulation v which are not on the triangulation E , and having at least the dyadics $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ on its boundary. Let a be an interior edge of the triangulation of P in v joining the dyadics p and q . Suppose that $p > q$. There exists a unique triangle in P having as vertices p, x, q , where x is a dyadic number $p > x > q$. We associate the triangle of E with vertices $\frac{1}{2}, 0, \frac{1}{4}$ to this triangle. Analogously, we associate to the triangle with vertices $y > p > q$ (or $p > q > y$) of P the triangle with vertices $\frac{3}{4}, 0, \frac{1}{2}$. Repeating this procedure finitely many times we construct an ideal polygon Q triangulated as E , whose sides are naturally associated to those of the polygon P . Then, the element $f \in \mathcal{T}$ defined by $f(Q) = P$ satisfies our claim.

For the second assertion it is enough to prove that given an element $f \in \mathcal{T}$ different from the identity, there exists a vertex v in \mathcal{C} such that $f \cdot v \neq v$. Let q be a dyadic such that $f(q) \neq q$ (it exists because $f \neq \text{id}$). Take a square inscribed in the triangulation E , having q as a vertex and such that it has empty intersection with its image by f . Now, let v be the triangulation obtained from E by flipping the diagonal of this square. Then, $f \cdot v \neq v$. \square

Proposition 5. *The complex \mathcal{C} is not Gromov hyperbolic.*

The proof uses the following observation:

Remark 7. Every flip changes a single arc of the initial triangulation. Therefore, the distance between two vertices $v, w \in \mathcal{C}$ is bounded below by the number of arcs which are on the triangulation v and are not in the triangulation w .

Proof. Let n be a positive integer. We construct a geodesic triangle u, v, w such that $d_{\mathcal{C}}(u, v) = d_{\mathcal{C}}(v, w) = n$, and $d_{\mathcal{C}}(u, w) = 2n$; and we give a point p in the geodesic segment uw such that $d_{\mathcal{C}}(p, x) \geq n$ for all $x \in uv \cup vw$.

Let s_1, \dots, s_{2n} be $2n$ disjoint squares inscribed in the triangulation E . Let $u = E$, v be the vertex obtained from E by flipping the diagonals of the squares s_1, \dots, s_n and w be the vertex obtained from v by flipping the diagonals of the squares s_{n+1}, \dots, s_{2n} . The segments uv and vw consist on flipping, respectively, the diagonals of s_1, \dots, s_n , and s_{n+1}, \dots, s_{2n} , in order. The segment uw consists on flipping first the diagonal of the square s_{2n} , then the diagonal of s_{2n-1} until the diagonal of s_1 , in decreasing order (see figure 3). The point p is the vertex obtained from E by flipping the diagonals of s_{n+1}, \dots, s_{2n} . By the remark 7 all these segments are geodesics, and $d_{\mathcal{C}}(p, x) = n + \min\{d_{\mathcal{C}}(u, x), d_{\mathcal{C}}(w, x)\}$, for all $x \in uv \cup vw$. \square

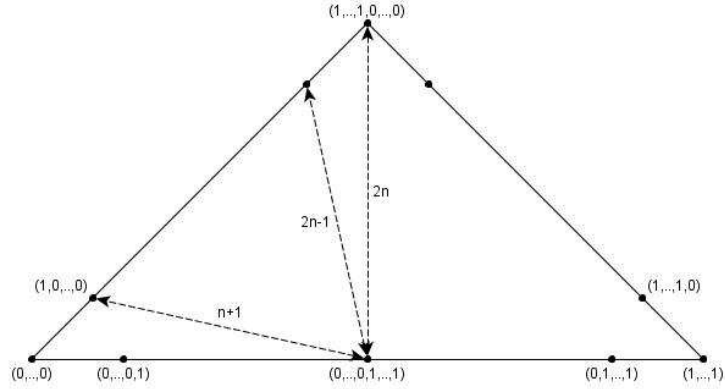


Figure 3: The points correspond to triangulations obtained from E by flipping the diagonals of s_i if there is a one in the i -th coordinate. Discontinuous lines represent distances in \mathcal{C} .

4 The automorphism group of the complex \mathcal{C}

In this section we study the group of automorphisms $\text{Aut}(\mathcal{C})$ of the asymptotic pants complex \mathcal{C} , and we prove that it is isomorphic to the semi-direct product $\mathcal{T} \rtimes \mathbb{Z}/2\mathbb{Z}$.

The proof of this result has some similarities with Ivanov's proof (see [14]) of the fact that the automorphism group of the arc complex of a compact surface is isomorphic to the extended mapping class group. In particular, he showed that the image of a maximal simplex determines completely an automorphism. In our case, every automorphism is determined by its image of the ball of radius one centred in some vertex v . Then, given an automorphism φ of the complex \mathcal{C} , we construct an element $t \in \Psi(\mathcal{T}) \subset \text{Aut}(\mathcal{C})$ such that $t(E) = \varphi(E)$. For this step, we need to introduce an auxiliary complex, which is related to the link of v in \mathcal{C} .

4.1 Construction of the link complex $\mathcal{L}^2(v)$

Recall that in any triangulation v of Σ' , the flips along two different arcs of v do not commute if and only if both arcs belong to the boundary of some triangle of v (remark 5). We construct a sub-complex of \mathcal{C} which encodes this information.

Let $\mathcal{L}^1(v)$ be a graph whose set of vertices is the set of vertices of \mathcal{C} adjacent to the vertex v . Two vertices are connected by an edge if they lie in the same pentagon in \mathcal{C} .

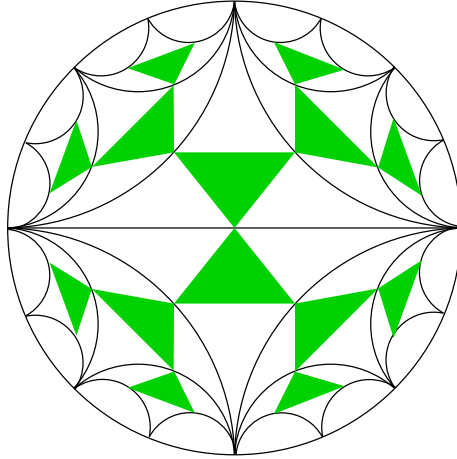


Figure 4: The link of E .

Given a vertex of \mathcal{C} there is a geometric interpretation of $\mathcal{L}^1(v)$: we associate to every arc a of the triangulation v a vertex which represents the vertex of \mathcal{C} obtained from v by flipping the arc a . Then, from the geometric interpretation of the 2-cells of \mathcal{C} (remark 5) we see that two vertices of $\mathcal{L}^1(v)$ associated to the arcs a, b are joined by an edge if and only if there is a triangle in v having both a, b on its boundary. Thus, $\mathcal{L}^1(v)$ can be seen as a connected union of triangles, each of them inscribed on a triangle of v , where the intersection between different triangles is either empty or a vertex, and where every vertex belong to two different triangles (see figure 4).

Definition 7. Let v be a vertex of \mathcal{C} . The link of v is the 2-dimensional complex $\mathcal{L}^2(v)$ obtained from $\mathcal{L}^1(v)$ by gluing a 2-cell on each triangle. Furthermore, the link lies on \mathbb{D}^2 and every triangle is naturally oriented.

Remark that the link has been constructed using only the combinatorial structure of \mathcal{C} in the neighbourhood of a vertex. Note also that \mathcal{C} is regular in the sense that all vertices have the same combinatorial structure on their neighbourhoods. Thus, one can derive the following result:

Lemma 2. Let v be a vertex of \mathcal{C} . Then, every automorphism $\phi \in \text{Aut}(\mathcal{C})$ induces an isomorphism $\phi_{*,v} : \mathcal{L}^2(v) \rightarrow \mathcal{L}^2(w)$, where $w = \phi(v)$.

Now, using the transitivity of the action of \mathcal{T} on \mathcal{C} one obtains:

Corollary 4. Let v and w be vertices of \mathcal{C} . Then, their links $\mathcal{L}^2(v)$ and $\mathcal{L}^2(w)$ are isomorphic.

4.2 Extensions of link isomorphisms

In the previous section we proved that every automorphism φ of \mathcal{C} induces link isomorphisms between $\mathcal{L}^2(v)$ and $\mathcal{L}^2(w)$, where $w = \varphi(v)$. Now, given two vertices v and w of \mathcal{C} and given an isomorphism $i : \mathcal{L}^2(v) \rightarrow \mathcal{L}^2(w)$ between their links, it is natural to ask under which conditions one can find an automorphism $\varphi \in \text{Aut}(\mathcal{C})$ such that $\varphi_{*,v} = i$. More specifically, which is the main obstruction to the existence of this automorphism?

Let $\varphi \in \text{Aut}(\mathcal{C})$. Let w be the image of v by φ . Let (a, b, c) be three vertices of a triangle Δ on the link $\mathcal{L}^2(v)$, and suppose they are cyclically ordered according to the orientation of the triangle. The vertices $\varphi(a), \varphi(b), \varphi(c)$ are the vertices of a triangle $\phi(\Delta)$ in the link $\mathcal{L}^2(w)$. Now, there are two possible cases:

1. The orientation of $\varphi(\Delta)$ induces the cyclic order $(\varphi(a), \varphi(b), \varphi(c))$ on its vertices. Then we say that φ is *orientation preserving on Δ* .

2. The orientation of $\varphi(\Delta)$ induces the cyclic order $(\varphi(a), \varphi(c), \varphi(b))$ on its vertices. Then we say that φ is *orientation reversing on Δ* .

Definition 8. Let $\varphi \in \text{Aut}(\mathcal{C})$ and v be a vertex of \mathcal{C} . We say that φ is *v -orientation preserving (reversing)* if it is orientation preserving (reversing) on every triangle of $\mathcal{L}^2(v)$.

Remark 8. Every element $f \in \mathcal{T}$ induces an automorphism $\Psi(f) \in \text{Aut}(\mathcal{C})$ (from proposition 4) which is v -orientation preserving for all vertices v of \mathcal{C} .

Lemma 3. Let v and w be vertices of \mathcal{C} , and $i : \mathcal{L}^2(v) \rightarrow \mathcal{L}^2(w)$ be an orientation preserving isomorphism of their links. Then, there exists a unique automorphism $\varphi_i \in \text{Aut}(\mathcal{C})$ such that $\varphi_{*,v} = i$. Furthermore, $\varphi_i \in \mathcal{T}$.

Proof. We first prove the existence by constructing $\varphi \in \mathcal{T}$ in the following way: let $\{I_{c_1}^{d_1}, \dots, I_{c_l}^{d_l}\}$ a set of separating arcs on the triangulation w containing all the edges which differ from E . Let u'_1, \dots, u'_l be the vertices of $\mathcal{L}^1(w)$ corresponding to the triangulations obtained from w by flipping, respectively, the edges $I_{c_1}^{d_1}, \dots, I_{c_l}^{d_l}$. Let $\{I_{a_1}^{n_1}, \dots, I_{a_k}^{n_k}\}$ be separating arcs on the triangulation v containing on their interior $i^{-1}(u'_1), \dots, i^{-1}(u'_l)$. Let u_1, \dots, u_k be the vertices of $\mathcal{L}^1(v)$ corresponding to the triangulations obtained from v by flipping, respectively, the edges $I_{a_1}^{n_1}, \dots, I_{a_k}^{n_k}$. Then, $i(u_1), \dots, i(u_k)$ are vertices corresponding to edges which form a set of separating arcs of w , $\{I_{b_1}^{m_1}, \dots, I_{b_k}^{m_k}\}$. Define φ by $\varphi(I_{a_j}^{n_j}) = I_{b_j}^{m_j}$ for $1 \leq j \leq k$. It is easy to see that $\varphi_{*,v} = i$.

Uniqueness: this proof is based on the remarks 5 and 6. Let $\psi \in \text{Aut}(\mathcal{C})$ such that $\psi_{*,v} = i$. Then, we prove that its action on the 0-skeleton of \mathcal{C} must coincide with φ .

- $\psi(v) = w$.
- For every element u on the unit sphere of \mathcal{C} centred in v , $\psi(u)$ must be $i(u)$.
- Suppose that ψ is defined in the ball $\mathcal{B}_{\mathcal{C}}(v, n)$ of \mathcal{C} of radius n centred in v for $n \geq 2$, and let u be a vertex at distance $n+1$ of v . Consider a path $p = v, u_1, \dots, u_{n-1}, u_n, u$ of length $n+1$ joining v and u . The vertices u_{n-1}, u_n, u define a unique 2-cell in \mathcal{C} (remark 6), and this 2-cell contains at least a fourth vertex $u' \in \mathcal{B}_{\mathcal{C}}(v, n)$. Thus, $\psi(u)$ must be the vertex of the unique 2-cell defined by $\psi(u'), \psi(u_{n-1}), \psi(u_n)$ which is adjacent to $\psi(u_n)$ and different from $\psi(u_{n-1})$.

□

Lemma 4. Let v and w be vertices of \mathcal{C} and let $i : \mathcal{L}^2(v) \rightarrow \mathcal{L}^2(w)$ be an isomorphism of their links such that it is orientation reversing on $\Delta_1 = (u_0, u_1, u_2)$ and orientation preserving on $\Delta_2 = (u_0, u_3, u_4)$. Then, there does not exist $\varphi \in \text{Aut}(\mathcal{C})$ with $\varphi_{*,v} = i$.

Proof. Suppose there exists $\varphi \in \text{Aut}(\mathcal{C})$ such that $\varphi_{*,v} = i$.

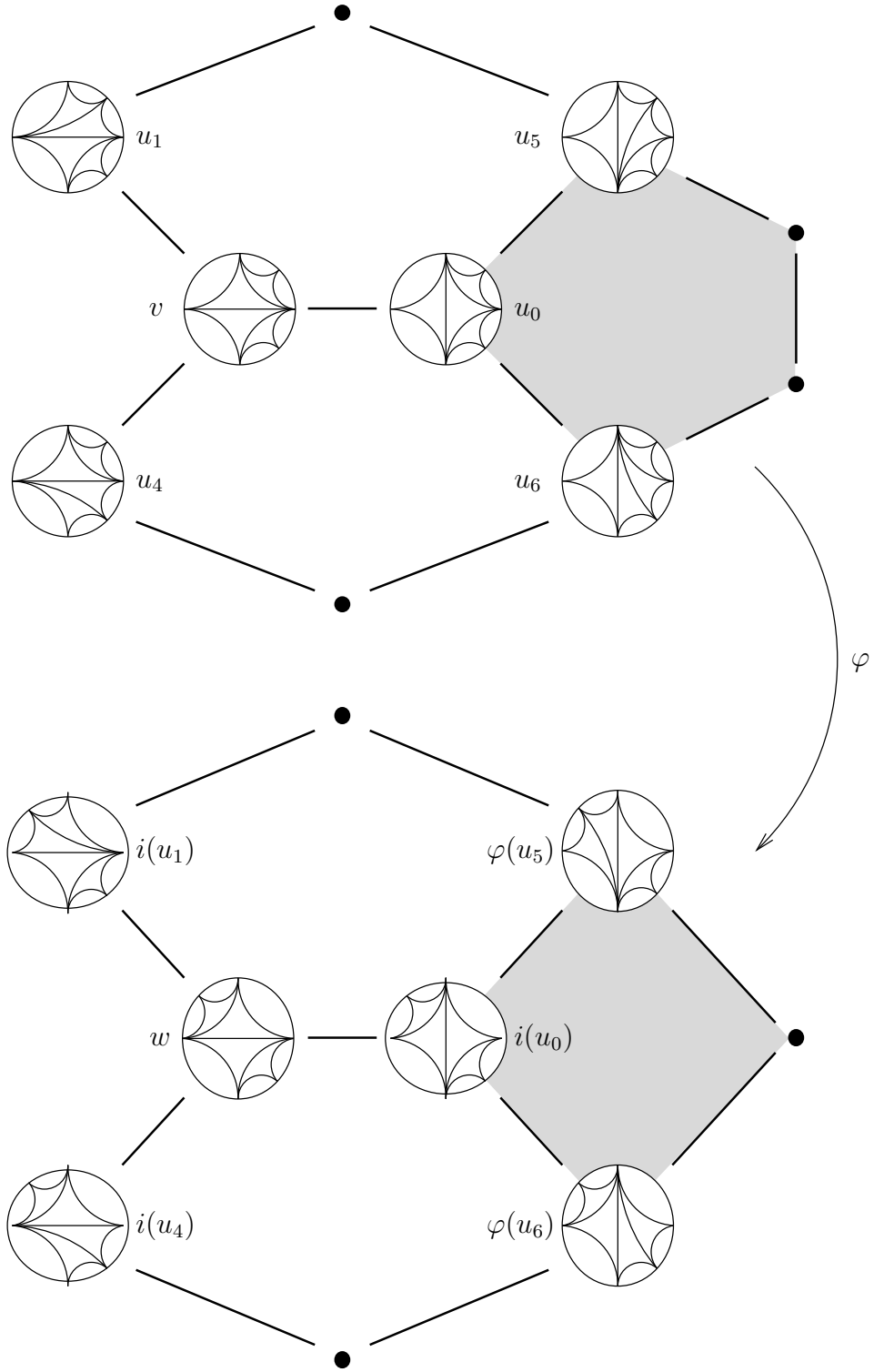


Figure 5: Example of lemma 4.

The vertices u_1, v, u_0 define a pentagonal 2-cell of \mathcal{C} ; let u_5 be the vertex at distance 1 of u_0 and different from v on this cell. Analogously, let u_6 be the vertex at distance 1 of u_0 different from v and lying on the pentagonal 2-cell defined by the path u_4, v, u_0 . The path u_5, u_0, u_6 defines a pentagonal 2-cell of \mathcal{C} . On the other hand, $i(u_1), i(v), i(u_0)$ also define a pentagonal 2-cell of \mathcal{C} . Adapting the prove of uniqueness

of the lemma 3, $\varphi(u_5)$ must be the vertex at distance 1 of $i(u_0)$ and different from w on this pentagonal 2-cell. Furthermore, $\varphi(u_6)$ must be the vertex at distance 1 of $\varphi(u_0)$ different from w and lying on the pentagonal 2-cell defined by the path $i(u_4), i(v), i(u_0)$. But in this case the path $\varphi(u_5), \varphi(u_0), \varphi(u_6)$ defines a squared 2-cell of \mathcal{C} , which contradicts the fact that $\varphi \in \text{Aut}(\mathcal{C})$. See figure 5 for a picture of the situation. \square

Remark 9. Let v be a vertex of \mathcal{C} and let w be one of its neighbours in \mathcal{C} . Let w, u_1, u_2 and w, u_3, u_4 denote the triangles of $\mathcal{L}^2(v)$ containing w . Observe that the geometric representation of the link $\mathcal{L}^2(w)$ of w has the same vertices as $\mathcal{L}^2(v)$ and only the triangles mentioned above change to w, u_1, u_4 and w, u_2, u_3 . See figures 4 and 6.

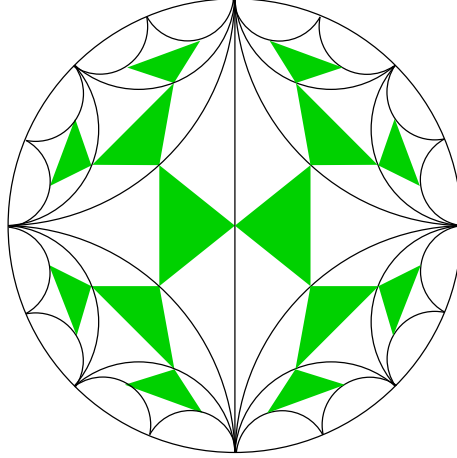


Figure 6: The link of the vertex of \mathcal{C} obtained from E by flipping I_0^1 .

This provides an alternative prove of lemma 4:

Proof. Suppose there exists $\varphi \in \text{Aut}(\mathcal{C})$ such that $\varphi_{*,v} = i$. Then φ_{*,u_0} must be an isomorphism between the link of u_0 and the link of $i(u_0)$. As a consequence of remark 9, u_1 and u_4 are adjacent in the link of u_0 , but $i(u_1)$ and $i(u_4)$ are not adjacent in the link of $i(u_0)$ (see figure 7), which contradicts the fact that φ_{*,u_0} is an isomorphism. \square

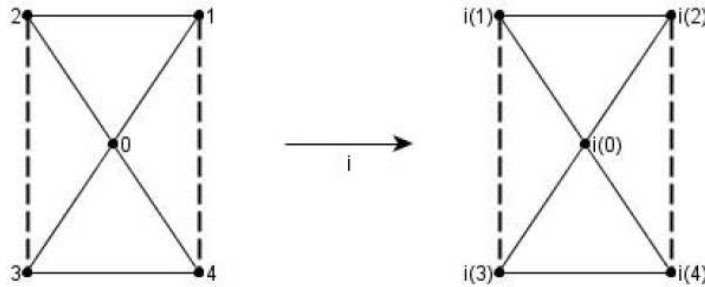


Figure 7: The map i in Δ_1 and Δ_2 . In discontinuous lines the links of u_0 and $i(u_0)$.

Lemma 5. Let $i_R : \mathcal{L}^2(E) \rightarrow \mathcal{L}^2(E)$ be the orientation reversing isomorphism obtained by the symmetry of axis I_0^1 . Then, there exists a unique automorphism $\varphi_R \in \text{Aut}(\mathcal{C})$ such that $\varphi_{R*,E} = i_R$.

Proof. The uniqueness can be proved exactly as in lemma 3. For the existence it suffices to send each vertex to its symmetric with respect to the symmetry of axis I_0^1 . \square

4.3 Proof of theorem 1

In order to simplify the prove of theorem 1 and give a geometric interpretation of it, we introduce the following concept:

Definition 9. Let $\varphi \in \text{Aut}(\mathcal{C})$. It is orientation preserving (reversing) if it is v -orientation preserving (reversing) for all vertices v of \mathcal{C} .

Remark 10. As a consequence of lemma 3, the subgroup of orientation preserving automorphisms of \mathcal{C} is isomorphic to \mathcal{T} .

Remark 11. The automorphism φ_R from lemma 5 is orientation reversing.

Using remarks 10 and 11, one obtains the following classification of the automorphisms of \mathcal{C} :

Proposition 6. Every automorphism of \mathcal{C} is either orientation preserving or orientation reversing.

Proof. As a consequence of lemma 4, given an automorphism φ of \mathcal{C} , its restriction $\varphi_{*,E}$ to the link $\mathcal{L}^2(E)$ is either E -orientation preserving or E -orientation reversing. In the first case, $\varphi \in \Psi(\mathcal{T})$ by lemma 3 and thus it is orientation preserving (remark 10). In the second case, consider the automorphism $\varphi' = \varphi \circ \varphi_R$, which is E -orientation preserving and thus orientation preserving. Then, $\varphi = \varphi' \circ \varphi_R$ (note that $\varphi_R^{-1} = \varphi_R$), therefore it is orientation reversing (remark 11). \square

Now, we prove theorem 1:

Proof. We want to construct the following exact sequence:

$$1 \longrightarrow \mathcal{T} \longrightarrow \text{Aut}(\mathcal{C}) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1,$$

and prove that it splits.

We denote by $\Pi : \text{Aut}(\mathcal{C}) \rightarrow \mathbb{Z}/2\mathbb{Z}$ and define it as follows:

$$\varphi \mapsto \begin{cases} 0, & \text{if } \varphi \text{ is orientation preserving} \\ 1, & \text{if } \varphi \text{ is orientation reversing.} \end{cases}$$

Proposition 6 shows that the map Π is a well defined morphism of groups. As a consequence of lemmas 3 and 5, the map Π is surjective and the isomorphism $\mathbb{Z}/2\mathbb{Z} \simeq \langle \varphi_R \rangle$ is a section of Π .

Finally, $\ker(\Pi) \simeq \mathcal{T}$ follows from lemma 3. \square

Therefore, we have an action of $\mathbb{Z}/2\mathbb{Z}$ into the group of automorphisms of \mathcal{T} , where the generator of $\mathbb{Z}/2\mathbb{Z}$ acts as the automorphism γ_R which is defined by $\gamma_R(t) = \varphi_R \circ t \circ \varphi_R$, for $t \in \mathcal{T}$. In particular, the automorphism γ_R obtained in \mathcal{T} is defined by $\gamma_R(\alpha) = \alpha^{-1}$ and $\gamma_R(\beta) = \alpha^2 \beta^{-1} \alpha^2$. Thus, if we compose γ_R with the conjugation by α^2 we obtain the generator of the group of outer automorphisms of \mathcal{T} sending α to α^{-1} and β to β^{-1} , which is the generator of $\text{Out}(\mathcal{T})$ given by Brin in [2].

Remark 12. (Geometric interpretation of theorem 1.) By proposition 6, all automorphisms of \mathcal{C} are either orientation preserving or orientation reversing. Thus, if one defines the *extended asymptotic mapping class group* of Σ as $\mathcal{T} \rtimes \mathbb{Z}/2\mathbb{Z}$, theorem 1 turns out to be an exact analogue to Ivanov's theorem for the planar surface of infinite type Σ .

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